CLOSED EXACT LAGRANGIANS IN THE SYMPLECTIZATION OF CONTACT MANIFOLDS

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Abstract. For a certain class of exotic contact manifolds of dimension greater than 3, we show that there is an abundance of closed exact Lagrangians in their symplectization. All of these Lagrangians are displaceable by Hamiltonian isotopy, and many of the examples are nulhomotopic.

Inside an exact symplectic manifold $(X, d\lambda)$ a *formal Lagrangian embedding* is a pair (f, Ψ_s), where $f : L \to X$ is a smooth embedding of a closed manifold L of dimension $n = \frac{1}{2}\dim X$, and $\Psi_s : TL \to TX$ is a homotopy through injective bundle homomorphisms covering f, so that $F_0 = df$ and F_1 is a map with Lagrangian image. The following theorem is a simple application of results from [1] and [4].

Theorem 1. *Let $(Y, \ker\alpha)$ be a contact manifold of dimension $2n-1 \geqslant 5$. Assume that Y contains a small plastikstufe with spherical core and trivial rotation. Let (f, Ψ_s) be a formal Lagrangian embedding of the closed manifold L into $(\mathbb{R}\times Y, d(e^t\alpha))$. Then f is isotopic to an exact Lagrangian embedding.*

First we cite a theorem from [4].

Proposition 2 ([4], Theorem 1.1)**.** *Let (Y, ξ) be a contact manifold which contains a small plastikstufe with spherical core and trivial rotation, denoted by $\mathcal{PS}$. Let $\Lambda \subseteq Y$ be a connected Legendrian submanifold which is disjoint from $\mathcal{PS}$. Then Λ is loose.*

Because the specifics are not relevant to our short paper, we will omit the definition of "small plastikstufe with spherical core and trivial rotation", as well as the definition of loose Legendrians. The plastikstufe was first defined and studied by Niederkrüger in [5], as an obstruction to symplectic fillability. Loose Legendrians were defined and classified up to Legendrian isotopy in [3]. Besides the original sources, an exposition of both topics is given in [4]. Loose Legendrians are useful for our construction due to the following theorem from [1]; to simplify the statement we restrict our citation to the case where the ambient manifold is a symplectization.

Proposition 3 ([1], Theorem 2.2)**.** *Let $(Y, \ker\alpha)$ be a contact manifold of dimension $2n-1$ and let (f, Ψ_s) be a formal Lagrangian embedding of the compact manifold L into $([0,\infty)\times Y, d(e^t\alpha))$, so that near a collar neighborhood of the boundary $[0,\epsilon)\times\partial L$ f is modeled on $f(t,p) = (t, g(p))$ for some Legendrian embedding $g : \partial L \to Y$ and Ψ_s is constant with respect to s. If $g(\partial L)$ is loose, then there is an isotopy of f which is fixed near ∂L to an exact Lagrangian embedding.*

Given a smooth isotopy $\varphi_t : X \to X$, we note that the composition $\varphi_1 \circ f$ can be made into a formal Lagrangian. This is done by taking the homotopy of

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bundle maps $d\varphi_{1-s} \circ F_s$ and translating them via a symplectic connection so that these maps cover $\varphi_1 \circ f$ for all s. This formal Lagrangian embedding is therefore non-canonical, but canonical up to homotopy.

We describe a simple model which will be useful for our construction. Inside $B^{2n} \subseteq \mathbb{C}^n$, consider the Lagrangian $L_0 = \{y_i = 0, i = 1, \dots n\}$. Notice that $\partial L_0 \subseteq S^{2n-1}$ is Legendrian with respect to the standard contact form $(\frac{1}{2}\sum_i y_i dx_i - x_i dy_i)|_{S^{2n-1}}$. We perurb L_0 by a Hamiltonian supported near the origin so that the origin is not contained in L_0. Choose a radius γ of B^{2n} which is disjoint from L_0. Then $B^{2n} \setminus \gamma$ is symplectomorphic to $((-\infty, 0] \times \mathbb{R}^{2n-1}, d(e^t \alpha_{\mathrm{std}}))$. Indeed, the negative Liouville flow on $B^{2n} \setminus \gamma$ is complete, and $\mathbb{R}^{2n-1}_{\mathrm{std}}$ is contactomorphic to S^{2n-1}_{std} with a single point removed. We say L_0 is the *standard Lagrangian filling* of the *standard Legendrian unknot*, ∂L_0.

Proof of Theorem 1: Choose a Darboux chart $U \subseteq Y$ which is disjoint from $\mathcal{PS}$. Let $\Lambda \subseteq U$ be the standard Legendrian unknot, and denote its standard Lagrangian filling by $L_0 \subseteq (-\infty, 0] \times U$. By smooth isotopy we can assume that $f(D) = L_0$ for some disk $D \subseteq L$ and we can then find a homotopy of Ψ_s so that on D, $\Psi_s = df$ for all $s \in [0, 1]$. We finally arrange by smooth isotopy that $f(L) \cap (-\infty, 0] \times Y = f(D)$, by finding a smooth isotopy which is fixed on $f(D)$ and moves the remainder of f in the positive t direction.

Because $\Lambda \subseteq U$ and U is disjoint from $\mathcal{PS}$, Proposition 2 tells us that Λ is loose. Proposition 3 tells us then that $f|_{L \setminus D}$ is isotopic to an exact Lagrangian embedding. Since f was already Lagrangian on D this completes the construction. □

Remark 4. Note that any exact Lagrangian in a symplectization is Hamiltonian displaceable, since any path of exact Lagrangian embeddings can be realized by Hamiltonian isotopy, and in a symplectization any t-shift of an exact Lagrangian is again an exact Lagrangian.

To make our result slightly more concrete, we point out that manifolds Y satisfying our hypotheses are plentiful. The construction is essentially due to Etnyre and Pancholi [2]; the proposition we cite is a slight modification.

Proposition 5 ([4], Proposition 5.2)**.** *Let (Y, ξ) be any contact manifold of dimension larger than* 3*, and let $U \subseteq Y$ be a Darboux chart. Then there is another contact structure ξ' on Y, so that $\xi' = \xi$ outside of U, and ξ' contains a small plastikstufe with spherical core and trivial rotation inside U. Furthermore ξ and ξ' are homotopic through almost contact structures.*

References

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